RAPID COMMUNICATIONS

PHYSICAL REVIEW E VOLUME 47, NUMBER 6 JUNE 1993

Counterpropagating periodic pulses in coupled Ginzburg-Landau equations

Boris A. Malomed*

Department of Physics, University of Illinois at Urbana-Champaign, 1110 West Green Street, Urbana, Illinois 61801

(Received 25 February 1993)

A recently observed stable regime in the form of periodically colliding counterpropagating wave packets (pulses) in an annular convection channel at very small positive overcriticalities is described analytically in terms of coupled Ginzburg-Landau equations. First, the existence of this regime is demonstrated in the framework of the simplest system including only the group-velocity difference, weak gain, and nonlinear dissipative coupling between two modes. In this approximation, the shape of the counterpropagating waves remains indefinite. It is demonstrated that additional dispersive terms, regarded as a small perturbation, provide shaping of the wave packets and also give rise to the deviation of the phase velocity from that for purely linear waves.

PACS number(s): 47.27.−i, 03.40.Kf

In recent experiments [1], Kolodner observed a stable regime of counterpropagation of two pulses (called wave packets in Ref. [1]) of convection in a narrow annular channel filled with a binary liquid heated from below. The experiments were conducted at positive but very small values of the overcriticality ϵ, $\epsilon=0.000\,18$–$0.001\,66$ (at larger ϵ, the so-called dispersive chaos [2] sets in). When far from each other, the wave packets undergo a slow growth due to the small overcriticality, which is compensated by losses during their collisions, so that a stable dynamical regime is observed with the mean velocity of the packets close to the group velocity of linear waves at the point $\epsilon=0$.

In Ref. [1], it was suggested that the regime observed could be directly interpreted in terms of the system of two coupled Ginzburg-Landau (GL) equations with complex coefficients governing envelopes of right- and left-propagating waves. For the first time, the system of coupled GL equations for counterpropagating waves was considered in Ref. [3] (however, in Ref. [3] only purely real coefficients were considered). The objective of the present work is to do this in the framework of the simplest systems of that type.

Usually, the GL equations with complex coefficients can be treated analytically in two cases: (i) when real parts of the coefficients are small in comparison to their imaginary parts, so that each equation is close to the nonlinear Schrödinger (NS) equation [4], or (ii) when the imaginary parts are small [5]. Recently, the interaction of two counterpropagating solitonlike pulses was considered in the near-NS regime, assuming weak coupling between the two equations [6]. In that work, a threshold (maximum) value of the relative group velocity admitting fusion of the colliding pulses into a bound state was found. Within the framework of the same approximation, it is, as a matter of fact, trivial to describe a regime similar to that reported in Ref. [1] (this would require one to assume that changes of the solitons' amplitude produced by a collision and by the slow growth between collisions are small enough). However, such a description does not seem relevant. It was emphasized in Ref. [1] that the pulses (wave packets) observed in the experiment looked quasilinear, thus being very different from true solitons.

The simplest dynamical model of the counterpropagating waves must include the group-velocity difference, linear gain, and nonlinear cross damping:

$$u_t+u_x=\epsilon u-|v|^2u \ , \tag{1a}$$

$$v_t-v_x=\epsilon v-|u|^2v \ , \tag{1b}$$

where the group velocities are chosen to be ±1. In spite of the simplicity of Eqs. (1), analytical solutions are not available in an exact form; nevertheless, a certain class of approximate solutions will be found below.

It is convenient to introduce new independent variables

$$\xi\equiv t+x,\tau\equiv t-x \tag{2}$$

and new unknown quantities

$$U\equiv|u|^2 \ , \quad V\equiv|v|^2 \ . \tag{3}$$

Then Eqs. (1) reduce to

$$U_\xi=\epsilon U-UV \ , \tag{4a}$$

$$V_\tau=\epsilon V-UV \ . \tag{4b}$$

Due to the symmetry between the right- and left-traveling waves, one should look for solutions satisfying the identity

$$U(\xi,\tau)\equiv V(\tau,\xi) \ . \tag{5}$$

In the case $\epsilon=0$, a general solution to Eqs. (4) and (5) can be readily found:

$$U_0(\xi,\tau)=g'(\tau)[g(\tau)+g(\xi)]^{-1} \ , \tag{6a}$$

$$V_0(\xi,\tau)=g'(\xi)[g(\tau)+g(\xi)]^{-1} \ , \tag{6b}$$

where g is an arbitrary function of one variable. According to Eqs. (3), only positive solutions for U and V are meaningful. Then it is necessary to select solutions periodic in x, to be able to model wave propagation in the annular channel. The simplest way to obtain solutions satifying these conditions is to take

$$g(z)=G(z)+kz\ , \tag{7}$$

where k is a constant, z stands for τ or ξ, and the function G is periodic. In terms of the original variables x and t, the corresponding solution (6) will be

$$U_0(x,t)=[k+G'(t-x)]\times[G(t-x)+G(t+x)+2kt]^{-1}\ , \tag{8a}$$

$$V_0(x,t)=[k+G'(t+x)]\times[G(t-x)+G(t+x)+2kt]^{-1}\ . \tag{8b}$$

Evidently, this solution is positive everywhere if k is sufficiently large. The terms in the denominators proportional to t describe slow attentuation of the waves produced by their periodic collisions in the absence of the gain ($\epsilon=0$).

If $\epsilon\neq 0$, a solution to Eqs. (4) is looked for in the form

$$U(\xi,\tau)=U_0(\xi,\tau)+\epsilon+U_1(\xi,\tau)\ , \tag{9a}$$

$$V(\xi,\tau)=V_0(\xi,\tau)+\epsilon+V_1(\xi,\tau)\ , \tag{9b}$$

where it is assumed that U_0 and V_0 are given by Eqs. (6) with some periodic function g, and U_1 and V_1 (but not ϵ) are small in comparison with U_0 and V_0. Then U_1 and V_1 are determined by the *linearized* equations following from insertion of Eqs. (9) into Eqs. (4):

$$(U_1)_\xi+V_0U_1+U_0V_1=-\epsilon V_0\ , \tag{10a}$$

$$(V_1)_\tau+V_0U_1+U_0V_1=-\epsilon U_0\ . \tag{10b}$$

To provide the assumed smallness of U_1 and V_1, one may take the function g in Eqs. (6) in the form

$$g(z)=1+h(K\epsilon z)\ , \tag{11}$$

where K is a large parameter, $K\gg 1$, and the periodic function h is small at all values of its argument, $|h|\ll 1$. Under these assumptions, the expressions (6) take the form

$$U_0\approx\tfrac{1}{2}K\epsilon h'(K\epsilon\tau)\ ,\quad V_0\approx\tfrac{1}{2}K\epsilon h'(K\epsilon\xi)\ . \tag{12}$$

Then it is easy to check that one may keep only the derivatives on the left-hand sides of Eqs. (10), so that these equations are integrated trivially:

$$U_1\approx\tfrac{1}{2}\epsilon h(K\epsilon\xi)\ ,\quad V_1\approx\tfrac{1}{2}\epsilon h(K\epsilon\tau)\ . \tag{13}$$

At last, comparing Eqs. (13) with Eqs. (12), one concludes that the smallness of U_1 and V_1 in comparison with U_0 and V_0 is provided by the large parameter K.

Thus the full solution (9) takes the form

$$U(\xi,\tau)=\tfrac{1}{2}[K\epsilon h'(K\epsilon\tau)-\epsilon h(K\epsilon\xi)]+\epsilon\ , \tag{14a}$$

$$V(\xi,\tau)=\tfrac{1}{2}[K\epsilon h'(K\epsilon\xi)-\epsilon h(K\epsilon\tau)]+\epsilon\ . \tag{14b}$$

Equations (14) represent a vast class of approximate periodic solutions of the underlying equations (4). Taking a particular solution, it is easy to specify conditions providing its positiveness. If, for example, one chooses $h(z)=h_0\cos z$, h_0 being a small constant accounting for the presumed smallness of $|h|$, the expressions (14) are positive, provided $h_0<2K^{-1}$. Note that in this case the minimum spatial period of the system is $L=2\pi/K\epsilon$. In a real situation, the period must be large, which can be provided by a choice of a sufficiently small ϵ. It is also pertinent to note that, actually, the first terms in the solution (14) represent noninteracting waves, and only the second terms give a small correction produced by the interaction.

So, the model based on Eqs. (1) gives rise to solutions describing steady counterpropagation of two symmetric periodic waves. A serious shortcoming of this model is, however, that it does not select any specific shape of the waves, while the shapes observed in the experiment [1] are well defined. Shaping can be provided by adding to Eqs. (1) dispersive (phase-modulation) terms and treating them as a small perturbation [7]:

$$u_t+u_x=\epsilon u-|v|^2u+i\alpha(|u|^2+\kappa|v|^2)u+i\beta u_{xx}\ , \tag{15a}$$

$$v_t-v_x=\epsilon v-|u|^2v+i\alpha(|v|^2+\kappa|u|^2)v+i\beta v_{xx}\ . \tag{15b}$$

Here α, $\alpha\kappa$, and β are the coefficients of the nonlinear self-phase modulation, nonlinear cross-phase-modulation, and spatial dispersion, respectively. Note that the values of α and β have been directly measured in experiments with the dispersive chaos [2], α being large and always positive, while β is small and may have either sign.

To take the dispersion terms into account as a small perturbation, one can look for a solution to Eqs. (15) in the form

$$u=u_0(\tau)\exp[i\omega t+i\phi(\tau,\xi)]\ ,\quad v=v_0(\xi)\exp[i\omega t+i\psi(\tau,\xi)]\ , \tag{16}$$

where u_0 and v_0 represent the unperturbed solution (14), in which the small terms are omitted and ω is a constant to be specified below. Insertion of Eqs. (16) into Eqs. (15) gives rise (in the lowest approximation) to the following equations for the phases ϕ and ψ:

$$2\phi_\xi=-\omega+\alpha\kappa v_0^2+\alpha u_0^2+\beta u_0^{-1}\frac{d^2u_0}{d\tau^2}\ , \tag{17a}$$

$$2\psi_\tau=-\omega+\alpha\kappa u_0^2+\alpha v_0^2+\beta v_0^{-1}\frac{d^2v_0}{d\xi^2}\ . \tag{17b}$$

According to the definition (2) of the variables ξ and τ, to provide the periodicity in x with some period L, it is necessary to look for solutions of Eqs. (17) that are periodic both in ξ and in τ with the same period. The periodicity condition applied to Eq. (17a) takes the form

$$\tfrac{1}{2}L\left[\beta u_0^{-1}\frac{d^2u_0}{d\tau^2}+\alpha u_0^2+\alpha\kappa\langle v_0^2\rangle-\omega\right]=2\pi n\ , \tag{18}$$

where $\langle\ \rangle$ stands for the mean value over the period, and $2\pi n$, with an arbitrary integer n, is a phase increment per period. The periodicity condition following from Eq. (17b) is an equation for the function v_0 symmetric to Eq. (18). Periodic solutions of Eq. (18) can be represented (for each set of values of ω and $\langle v_0^2 \rangle$) as a one-parametric family of Jacobian elliptic functions. Actually, the free parameter (elliptic modulus) is selected by the given period L. Next, coming back to Eqs. (14) (recall that the small terms in those equations are now omitted, and $U \equiv u_0^2$, $V \equiv v_0^2$), one immediately concludes that

$$\langle u_0^2 \rangle = \langle v_0^2 \rangle = \epsilon \ . \tag{19}$$

Equation (19) determines the value of the constant $\langle v_0^2 \rangle$ in Eq. (18), and it simultaneously imposes an additional constraint on solutions of Eq. (18), which allows one to determine the constant ω. In the case when the period L is large enough and $\alpha/\beta > 0$, the corresponding Jacobian function amounts to a rarefied chain of "solitons." As follows from Eqs. (18) and (19), each "soliton" in the chain and the corresponding value of ω can be represented in the form

$$u_0(\tau) = \sqrt{2\beta/\alpha} a / \cosh(a\tau) \ , \tag{20}$$

$$a \equiv \tfrac{1}{4} L (\alpha/\beta) \epsilon \ , \tag{21}$$

$$\omega = \beta a^2 + \alpha \kappa \epsilon - 4\pi \beta L^{-1} n \tag{22}$$

[note that for sufficiently large L the first term on the right-hand side of Eq. (22) is much larger than two others]. The shape of the pulses (wave packets) observed in Ref. [1] seems close to that given by Eq. (20). Equations (20) and (21) predict that the amplitude of the pulses, a, must be a linear function of ϵ. In fact, the data reported in Ref. [1] suggest that the amplitude is proportional to $\sqrt{\epsilon}$ rather than to ϵ. This discrepancy may be produced by the additional dissipative terms proportional to u_{xx} and v_{xx}, which were not included into Eqs. (15). It is known that the value of the coefficient in front of the corresponding term in the effective GL equation for the real traveling-wave convection is much larger than the value of the parameter β in Eqs. (15) [2]. It is also relevant to note that in the experiments reported in Ref. [1] the wave packets (pulses) observed were, generally speaking, rather wide, and only at the highest value of ϵ dealt with could one certainly say that the pulse occupied a spatial domain essentially smaller than the entire system. This circumstance may contribute to the discrepancy with the quantitative predictions deduced for the "solitons" [7].

Comparing Eqs. (20) and (21) with the general solution (14), one notices that in the present case $K = \frac{1}{4} L(\alpha/\beta)$. For the real traveling-wave convection [2], $\alpha \sim 8$, $\beta \sim 0.02$, so that the condition $K \gg 1$, which is necessary for the applicability of the approximate solution (14), can be satisfied even for small L.

When the GL equations are derived from the underlying "microscopic" physical equations, they appear as equations for *envelopes* of the traveling waves. Therefore, the additional frequency ω in Eqs. (16) gives rise to additional phase velocities $\pm V_{\mathrm{ph}} \sim \pm \omega$ of the right- and left-traveling waves. The change of the phase velocities with the growth of ϵ was observed in Ref. [1].

In conclusion, the analysis developed in the present work makes it possible to explain, at least qualitatively, the steady counterpropagation of two pulses in terms of coupled GL equations. Probably, extensive numerical simulations are necessary to attain a better quantitative agreement with the experiment.

I am indebted to Paul Kolodner for a copy of Ref. [1] and for a stimulating discussion. The hospitality of the Clarkson University (Potsdam, NY), and of the University of Illinois at Urbana-Champaign, where a part of this work was done, is appreciated.

*Permanent address: Department of Applied Mathematics, School of Mathematical Sciences, Tel Aviv University, Ramat Aviv 69978, Israel; electronic address: malomed@leo.math.tau.ac.il

[1] P. Kolodner, Phys. Rev. Lett. **69**, 2519 (1992).
[2] P. Kolodner, J. A. Glazier, and H. L. Williams, Phys. Rev. Lett. **65**, 1579 (1990); J. A. Glazier, P. Kolodner, and H. Williams, J. Stat. Phys. **64**, 945 (1991).
[3] M. C. Cross, Phys. Rev. A **38**, 3593 (1988).
[4] B. A. Malomed and A. A. Nepomnyashchy, Phys. Rev. A **42**, 6009 (1990).
[5] B. A. Malomed, Phys. Rev. A **44**, 6954 (1991).
[6] B. A. Malomed, Physica A **188**, 201 (1992).
[7] P. Kolodner (private communication).